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An Improvement of the Fischer Inequality

Russell Merris*

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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An improvement of the classical Fischer inequality for the determinant of a positive definite hermitian matrix is proved. It is used to analyze the Hadamard determinant theorem.

Key words: Fischer inequality; Hadamard determinant theorem; permanent; positive definite hermitian matrix.

1. Introduction

Let $H=(h_{ij})$ be an n-square positive semidefinite hermitian matrix, $n \geqslant 2$. The justly famous Hadamard determinant theorem [3; 1, p. 64; 5, p. 114][1] states that

$$\prod_{t=1}^{n} h_{tt} \geqslant \det H.$$

Two of the many generalizations of this theorem are of interest to us.

Partition H,

$$H=\begin{bmatrix} H_{11} & H_{12} \\ H_{12}^{*} & H_{22} \end{bmatrix} \tag{1}$$

so that H_{22} is k square. E. Fischer [2; 5, p. 117–118] proved that

$$\det H_{11} \det H_{22} \geqslant \det H.$$

One obtains Hadamard's theorem from Fischer's by induction.

In 1918, I. Schur [7] extended Fischer's inequality. He defined certain matrix functions: Let G be a subgroup of the symmetric group S_n. Suppose λ is a character of degree s on G. Then

$$d(H) \equiv \sum_{\sigma \epsilon G} \lambda(\sigma) \prod_{t=1}^{n} h_{t\sigma(t)}.$$

If $G=S_n$ and $\lambda=\mathrm{sgn}$, $d=\det$. If $G=S_n$ and $\lambda \equiv 1$, $d=\mathrm{per}$ (permanent). Finally, if $G=\{1\}$ then $d(H)=s\prod_{t=1}^{n} h_{tt}$. Schur proved that

$$d(H) \geqslant s \det(H).$$

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*This work was done while the author was a National Academy of Sciences-National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234.

[1] Figures in brackets indicate the literature references at the end of this paper.

If S'_k is the symmetric group on $\{n-k+1, \ldots, n\}$, $G=S_{n-k}\times S'_k$, and $\lambda=\text{sgn}$, one obtains Fischer's inequality.

In this note an inequality improving Fischer's is derived. Though the methods are simple, the result allows a sharp analysis of Hadamard's theorem.

2. Results

THEOREM 1: *Let* H *be a positive definite hermitian matrix partitioned as in* (1). *Then*

$$\det H_{11} \det H_{22} - \det H \geqslant \det H_{11} \det (H_{12}^* H_{11}^{-1} H_{12})$$

with equality when $k=1$.

When $n=2k$, the result becomes

$$\det H_{11} \det H_{22} - \det H_{12}^* \det H_{12} \geqslant \det H,$$

a special case of Schur's theorem.

Before proving Theorem 1, we introduce the *Schur complement* of H_{11} in H as

$$(H/H_{11}) \equiv H_{22} - H_{12}^* H_{11}^{-1} H_{12}.$$

LEMMA (known): *If* H *is positive definite hermitian, so is* (H/H_{11}); *and* $\det H = \det H_{11} \det (H/H_{11})$.

PROOF: Haynsworth [4] has noticed that

$$Q^*HQ = \begin{bmatrix} H_{11} & 0 \\ 0 & (H/H_{11}) \end{bmatrix} \text{ where } Q = \begin{bmatrix} I_{n-k} & -H^{-1}H_{12} \\ 0 & I_k \end{bmatrix}.$$

Hence, (H/H_{11}) is positive definite because the nonsingular congruence, Q^*HQ, preserves positive definiteness. The determinant formula (due originally to Schur) follows because $\det Q=1$.

PROOF of THEOREM 1: It is a well known and elementary fact that if A and B are positive semidefinite hermitian matrices of the same size then

$$\det (A+B) \geqslant \det A + \det B.$$

(In fact, a much stronger result is known, namely the Minkowski determinant theorem [5, p. 115].) Using this fact, the definition of (H/H_{11}), and the lemma, we see

$$\det H_{22} \geqslant \det (H/H_{11}) + \det H_{12}^* H_{11}^{-1} H_{12}. \tag{2}$$

(Clearly, (2) is equality when $k=1$.) Multiplication of (2) by $\det H_{11}$ yields the result.

We proceed to use Theorem 1 to analyze Hadamard's theorem.

THEOREM 2: *Let* H *be positive definite hermitian. Suppose* u_t *is the row vector* $(h_{t,1}, \ldots, h_{t,t-1})$, *and* H_t *is the leading* $t\times t$ *principal submatrix of* H. *Then*

$$h_{tt} - u_t H_{t-1}^{-1} u_t^* > 0$$

and

$$h_{11} \prod_{t=2}^{n} (h_{tt} - u_t H_{t-1}^{-1} u_t^*) = \det H.$$

In a certain sense, Theorem 2 gives the final word on the Hadamard theorem. It shows just what is being thrown away to obtain the inequality.

PROOF: The proof is by induction on n. From Theorem 1 (with $k=1$),

$$\det H_{n-1}(h_{nn}-u_nH_{n-1}^{-1}u_n^*)=\det H.$$

But, $\det H_{n-1}$, $\det H>0$. Hence $(h_{nn}-u_nH_{n-1}^{-1}u_n^*)>0$. By induction,

$$\det H_{n-1}=h_{11}\prod_{t=2}^{n-1}(h_{tt}-u_tH_{t-1}^{-1}u_t^*)$$

and $(h_{tt}-u_tH_{t-1}^{-1}u_t^*)>0$.

(Alternatively, one can prove, using the Laplace expansion theorem [5, p. 14], that

$$\frac{\det H_t}{\det H_{t-1}}=h_{tt}-u_tH_{t-1}^{-1}u_t^*.)$$

Using different methods, Marcus and Soules [6] obtained Theorem 1 for the case $k=1$. They also obtained an analog of it for the permanent. They used their result to prove Theorems 3 and 4 (below) but failed to notice Theorems 2 and 5.

THEOREM 3: *Let* H *be positive definite hermitian. Let* u_t *and* H_t *be as in Theorem* 2. *Then*

$$\prod_{t=1}^{n}h_{tt}-\det H=(\det H_{n-1})u_nH_{n-1}^{-1}u_n$$

$$+\sum_{k=2}^{n-1}\left(\prod_{j=n-k+2}^{n}h_{jj}\right)(\det H_{n-k})(u_{n-k+1}H_{n-k}^{-1}u_{n-k+1}^*).$$

The proof is an easy induction on n.

THEOREM 4: *Let* H *be positive definite hermitian with minimum eigenvalue* μ. *Then*

$$\prod_{t=1}^{n}h_{tt}-\det H\geqslant\mu^{n-2}\sum_{i<j}|h_{ij}|^2.$$

PROOF: Let ρ_{n-k} be the largest eigenvalue of H_{n-k}. Cauchy's inequalities [5, pp. 119] yield

(i) $\prod_{t=n-k+2}^{n}h_{tt}\geqslant\mu^{k-1}$,

(ii) $\det H_{n-k}\geqslant\rho_{n-k}\mu^{n-k-1}$,

(iii) $u_{n-k+1}H_{n-k}^{-1}u_{n-k+1}^*\geqslant\rho_{n-k}^{-1}\|u_{n-k+1}\|^2$.

The result follows from (i), (ii), (iii), and Theorem 3.

THEOREM 5: *Let* H *be positive definite hermitian with largest eigenvalue* ρ. *Then*

$$\prod_{t=1}^{n}h_{tt}-\det H\geqslant\frac{\det H}{\rho^2}\sum_{i<j}|h_{ij}|^2. \tag{3}$$

PROOF: For $1\leqslant k\leqslant n$, let $H(k)$ denote the submatrix of H obtained by deleting row and column k. Then

(i) $\left(\prod_{t=n-k+2}^{n} h_{tt}\right) \det H_{n-k} \geqslant \det H(n-k+1)$,

(ii) $\det H(n-k+1) \geqslant \rho^{-1} \det H$,

(iii) $u_{n-k+1} H_{n-k}^{-1} u_{n-k+1}^{*} \geqslant \rho^{-1} \|u_{n-k+1}\|^2$.

Number (i) follows by induction on Fischer's inequality or from Schur's theorem. Numbers (ii) and (iii) follow from Cauchy's inequalities. The result follows from (i), (ii), (iii), and Theorem 3.

Theorem 5 can be restated as

THEOREM 5′: *Let* $\lambda_1, \ldots, \lambda_n$ *be the eigenvalues of the positive definite matrix* H. *Then*

$$\prod_{t=1}^{n} \left(\frac{h_{tt}}{\lambda_t}\right) - 1 \geqslant \frac{1}{2\rho^2} \sum_{t=1}^{n} (\lambda_t^2 - h_{tt}^2) \geqslant 0.$$

PROOF: Divide (3) by $\det H = \lambda_1 \cdots \lambda_n$ and observe that

$$2 \sum_{i<j} |h_{ij}|^2 + \sum_{i=1}^{n} h_{ii}^2 = \operatorname{trace} H^2 = \sum_{i=1}^{n} \lambda_i^2.$$

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Note Added in Proof: It has been pointed out to me that theorem 1 was proved by Frederic T. Metcalf, A Bessel-Schwarz inequality for Gramiams and related bounds for determinants, Ann. Mat. Pura ed Appl. **68,** 201–232 (1965), corollary 10.2.

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